LECTURE: 5-4 INDEFINITE INTEGRALS AND THE NET CHANGE THEOREM

Question: What does it mean to say $F(x)$ is an anti-derivative of $f(x)$? In other words, how would you check that $\int f(x)dx = F(x)$?

Check whether $\frac{d}{dx} F(x) = f(x)$.

If you differentiate your "answer" and get back the function in the integrand (integral sign) you have found an anti-deriv.)

Example 1: Verify by differentiation that $\displaystyle\int x\cos x dx = x\sin x + \cos x + C$ is correct.

$\frac{d}{dx}(x\sin x + \cos x + c) = 1\sin x + x\cos x - \sin x + 0$

$= x\cos x$ ✓

Indefinite Integrals

$$\int f(x)dx = F(x) \quad \text{means} \quad F'(x) = f(x)$$

All the indefinite integrals you (should) already know:

- $\int x^n dx = \boxed{\frac{1}{n+1}x^{n+1} + c}$ $\boxed{n \neq -1}$
- $\int \sin x dx = \boxed{-\cos x + c}$
- $\int \cos x dx = \boxed{\sin x + c}$
- $\int \sec^2 x dx = \boxed{\tan x + C}$
- $\int \csc^2 x dx = \boxed{-\cot x + c}$
- $\int \sec x \tan x \, dx = \boxed{\sec x + c}$
- $\int \csc x \cot x \, dx = \boxed{-\csc x + c}$
- $\int \frac{1}{x} dx = \boxed{\ln|x| + C}$
- $\int e^x dx = \boxed{e^x + C}$
- $\int a^x dx = \boxed{\frac{1}{\ln a} a^x + c}$ b/c $\frac{d}{dx} a^x = \ln a \, a^x$
- $\int \frac{1}{\sqrt{1-x^2}} dx = \boxed{\sin^{-1} x + C}$
- $\int \frac{1}{1+x^2} dx = \boxed{\tan^{-1} x + c}$

means include +c

Example 2: Find the general (what does "general" mean here?) indefinite integrals:

(a) $\int (10x^4 - 2\sec^2 x + \pi)dx$

$= 10 \cdot \frac{1}{5} x^5 - 2\tan x + \pi x + c$

$= \boxed{2x^5 - 2\tan x + \pi x + c}$

(b) $\int (x+1)(1+2x^4)dx = \int (x + 1 + 2x^5 + 2x^4)dx$

$= \frac{1}{2}x^2 + x + \frac{2}{6}x^6 + \frac{2}{5}x^5 + c$

$= \boxed{\frac{1}{3}x^6 + \frac{2}{5}x^5 + \frac{1}{2}x^2 + x + c}$

Example 3: Find the general indefinite integral:

(a) $\int \frac{\cos x}{1-\cos^2 x}dx = \int \frac{\cos x}{\sin^2 x}\,dx$

Know: $\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$= \int \frac{\cos x}{\sin x}\,\frac{1}{\sin x}\,dx$

$= \int \cot x \csc x\,dx$

$= \boxed{-\csc x + C}$

(b) $\int (x^2 + 2^x + 1)dx$

$= \boxed{\frac{1}{3}x^3 + \frac{1}{\ln 2}\cdot 2^x + x + C}$

Example 4: Find the following indefinite integrals.

a) $\int \left(\frac{3-x}{x}\right)^2 dx = \int \frac{(3-x)(3-x)}{x^2}\,dx$

$= \int \frac{9-6x+x^2}{x^2}\,dx$

$= \int \frac{9}{x^2} - \frac{6}{x} + 1 \quad dx$

$= \int (9x^{-2} - 6/x + 1)\,dx$

$= \frac{9x^{-1}}{-1} - 6\ln|x| + x + C$

$= \boxed{-9/x - 6\ln|x| + x + C}$

b) $\int \left(\frac{x}{7} - \frac{7}{x}\right) dx = \int \left(\frac{1}{7}x - 7\cdot\frac{1}{x}\right)dx$

$= \frac{1}{7}\cdot\frac{1}{2}x^2 - 7\ln|x| + C$

$= \boxed{\frac{1}{14}x^2 - 7\ln|x| + C}$

def. integrals +c cancels

Example 5: Evaluate the following integrals. Why is the $+C$ unnecessary here?

(a) $\int_0^9 \sqrt{2x}dx = \sqrt{2}\int_0^9 \sqrt{x}\,dx$

$= \sqrt{2}\int_0^9 x^{1/2}\,dx$

$= \left(\sqrt{2}\,\frac{2}{3}x^{3/2} + C\right)\Big|_0^9$

$= \frac{2\sqrt{2}}{3}\left(9^{3/2} + C - (0^{3/2} + C)\right)$

$= \frac{2\sqrt{2}}{3}\cdot 27$

$= \boxed{18\sqrt{2}}$

(b) $\int_0^1 x(\sqrt[4]{x^5} + \sqrt[5]{x})dx = \int_0^1 x(x^{5/4} + x^{1/5})dx$

$= \int_0^1 (x^{9/4} + x^{6/5})\,dx$

$= \left(\frac{4}{13}x^{13/4} + \frac{5}{11}x^{11/5}\right)\Big|_0^1$

$= \frac{4}{13}\frac{11}{11} + \frac{5}{11}\frac{13}{13}$

$= \frac{44+65}{143}$

$= \boxed{\frac{109}{143}}$

Example 6: Evaluate the following integrals.

(a) $\displaystyle\int_1^9 \frac{2t^2+t^2\sqrt{t}-1}{t^2}dt = \int_1^9 \frac{2t^2}{t^2}+\frac{t^2\sqrt{t}}{t^2}-\frac{1}{t^2}\,dt$

$$= \int_1^9 (2 + t^{1/2} - t^{-2})\,dt$$

$$= \left(2t + \frac{2}{3}t^{3/2} - \frac{t^{-1}}{(-1)}\right)\Bigg|_1^9$$

$$= \left(18 + \frac{2}{3}(27) + \frac{1}{9}\right) - \left(2 + 2/3 + 1\right)$$

$$= 18 + 18 + 1/9 - 3 - 2/3$$

$$= 33\ 9/9 + 1/9 \quad - 2/3\ 3/3$$

$$= \frac{297}{9} + \frac{1}{9} - \frac{6}{9} = \boxed{\frac{292}{9}}$$

(b) $\displaystyle\int_{\pi/4}^{\pi/2} \frac{1+\sin^2\theta}{\sin^2\theta}d\theta = \int_{\pi/4}^{\pi/2} \frac{1}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta}\,d\theta$

$$= \int_{\pi/4}^{\pi/2} (\csc^2\theta + 1)\,d\theta$$

$$= (-\cot\theta + \theta)\Big|_{\pi/4}^{\pi/2}$$

$$= -\frac{\cos\pi/2}{\sin\pi/2} + \pi/2 - \left(-\frac{\cos\pi/4}{\sin\pi/4} + \pi/4\right)$$

$$= 0 + \pi/2 + 1 - \pi/4$$

$$= \boxed{1 + \pi/4}$$

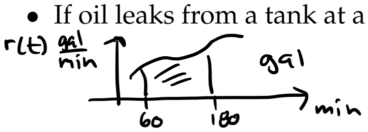
The Net Change Theorem

The integral of a rate of change is the net change:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Examples of Physical Situations

- If oil leaks from a tank at a rate of $r(t)$ gallons per minute at time t, what does $\displaystyle\int_{60}^{180} r(t)dt$ represent?

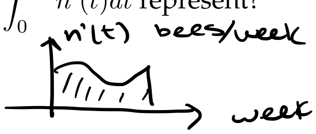


$\boxed{\text{\# of gallons leaked from } t=60 \text{ to } t=180}$

- A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \displaystyle\int_0^{15} n'(t)dt$ represent?

population (start) + growth

$= \boxed{\text{total population at 15 weeks}}$



- If $w'(t)$ is rate of growth of a child in pounds per year, what does $\displaystyle\int_0^5 w'(t)dt$ represent?

weight gained by the child at 5 weeks. To get the weight of the child you must add weight at birth.

- If the units for x are feet and $a(x)$ are pounds per foot what are the units for the following?

(a) $\dfrac{da}{dx}$ $\quad \dfrac{\text{lbs/ft}}{\text{ft}} = \boxed{\text{lbs/ft}^2}$

(b) $\displaystyle\int_2^8 a(x)dx$ $\quad \dfrac{\text{lbs}}{\text{ft}} \cdot \text{ft} = \boxed{\text{lbs}}$

Example 7: The water flows from the bottom of a storage tank at a rate of $r(t) = 500 - 2t$ gallons per minute for $0 \le t \le 250$. Find the total amount of water that flows from the tank during the first hour.

$$\text{Total } H_2O = \int_0^{250} r(t)\,dt$$

$$= \int_0^{250} (500 - 2t)\,dt$$

$$= (500t - t^2)\Big|_0^{250}$$

$$= 500(250) - 250^2$$

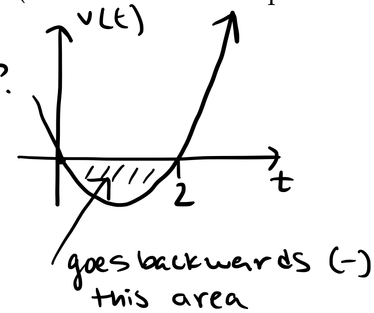
$$= 125000 - 62500$$

$$= \boxed{62500 \text{ gal}}$$

Example 8: A particle moves along a line so that its velocity at time t is $v(t) = t^2 - 2t$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \le t \le 4$.

How much has it moved relative to start?



$$\text{displacement} = \int_1^4 (t^2 - 2t)\,dt$$

$$= \left(\tfrac{1}{3}t^3 - t^2\right)\Big|_1^4$$

$$= \frac{64}{3} - 16 - \left(\frac{1}{3} - 1\right)$$

$$= \frac{63}{3} - 15$$

$$= 21 - 15 = \boxed{6 \text{ meters forward}}$$

(b) Find the distance traveled during this time period.

$$\text{dist traveled} = \int_1^4 |(t^2 - 2t)|\,dt$$

$$= -\int_1^2 (t^2 - 2t)\,dt + \int_2^4 (t^2 - 2t)\,dt$$

$$= \int_1^2 (2t - t^2)\,dt + \int_2^4 (t^2 - 2t)\,dt$$

$$= \left(t^2 - \tfrac{1}{3}t^3\right)\Big|_1^2 + \left(t^3/3 - t^2\right)\Big|_2^4$$

$$= \left(4 - \tfrac{8}{3}\right) - \left(1 - \tfrac{1}{3}\right) + \left(\tfrac{64}{3} - 16\right) - \left(\tfrac{8}{3} - 4\right)$$

$$= 4 - 1 - 16 + 4 - \tfrac{8}{3} + \tfrac{1}{3} + \tfrac{64}{3} - \tfrac{8}{3}$$

$$= -9 + \tfrac{49}{3} = -\tfrac{27}{3} + \tfrac{49}{3} = \boxed{\tfrac{22}{3}}\ \boxed{\text{m}}$$

$$\begin{array}{r} \overset{5}{\cancel{6}}5 \\ -16 \\ \hline 49 \end{array}$$